

# Some compatible Poisson structures and integrable bi-Hamiltonian systems on four dimensional and nilpotent six dimensional symplectic real Lie groups

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## Abstract

We give a new method for obtaining of compatible Poisson structures on Lie groups by means of the adjoint representations of Lie algebras. In this way we calculate some compatible Poisson structures on four dimensional and nilpotent six dimensional symplectic real Lie groups. Then using Magri-Morosi's theorem we obtain new bi-Hamiltonian systems with four dimensional and nilpotent six dimensional symplectic real Lie groups as phase spaces.

**keywords:** Integrable bi-Hamiltonian system, Compatible Poisson structures, Symplectic Lie group.

## 1 Introduction

Bi-integrable systems are interesting examples among integrable systems. First example of bi-integrable systems are bi-Hamiltonian systems. The study of integrable bi-Hamiltonian systems have been started with the pioneering work of Magri [1] and developed later in many papers (see for example [2], [3] and [4]). The bi-Hamiltonian structure has been observed in many of classical systems and some new interesting examples of bi-Hamiltonian systems have been discovered (see for example [5], [6] and [7]). In this work, we give a new method to construct compatible Poisson structures on a Lie group by means of the adjoint representation of its Lie algebra and construct integrable bi-Hamiltonian systems by using Magri-Morosi's theorem [8] (for a review see [9]). We give a method to produce integral of motion of a non-degenerate bi-Hamiltonian systems, for which the Lie group is the phase space.

The outline of the paper is as follows. In section two after reviewing the definition of compatible Poisson structures we give a new method for calculating of compatible Poisson structures on a Lie group (in general). Then in section three we obtain these structures on symplectic four dimensional real Lie groups. In section four using Magri-Morosi's theorem we obtain new bi-Hamiltonian system on symplectic four dimensional real Lie groups (as phase space). In the same way we obtain the compatible Poisson structures on symplectic nilpotent six dimensional real Lie groups [10] (see [11] for a rigorous commutative relations) in section five, of course in this section we obtain vierbeins on symplectic nilpotent six dimensional real Lie groups, where the results are summarized in appendix B. Finally in section six some new bi-Hamiltonian systems on symplectic nilpotent six dimensional real Lie groups are obtained. Some concluding remarks are given in section seven. The list of four dimensional and nilpotent six dimensional symplectic real Lie algebras are given in appendix A and B

## 2 Compatible Poisson structures and bi-Hamiltonian systems on Lie groups

### 2.1 Definitions and Notations

For self containing of the paper let us have a short review on the compatible Poisson structures and bi-Hamiltonian system (for a review see [9]).

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**Definition**[8]: A pair of Poisson brackets  $\{.,.\}$  and  $\{.,.\}'$  or a pair of Poisson bivectors  $\mathbf{P}$  and  $\mathbf{P}'$  on an  $m$  dimensional manifold  $\mathbf{M}$  is called compatible if we have:

$$[\mathbf{P}, \mathbf{P}] = [\mathbf{P}', \mathbf{P}'] = [\mathbf{P}, \mathbf{P}'] = 0, \quad (1)$$

where  $[.,.]$  is the Schoutn bracket that have the following forms:

$$[\mathbf{P}, \mathbf{P}]^{\lambda\mu\nu} = \mathbf{P}^{\rho\lambda} \partial_\rho \mathbf{P}^{\mu\nu} + \mathbf{P}^{\rho\nu} \partial_\rho \mathbf{P}^{\lambda\mu} + \mathbf{P}^{\rho\mu} \partial_\rho \mathbf{P}^{\nu\lambda}, \quad (2)$$

$$[\mathbf{P}', \mathbf{P}']^{\lambda\mu\nu} = \mathbf{P}'^{\rho\lambda} \partial_\rho \mathbf{P}'^{\mu\nu} + \mathbf{P}'^{\rho\nu} \partial_\rho \mathbf{P}'^{\lambda\mu} + \mathbf{P}'^{\rho\mu} \partial_\rho \mathbf{P}'^{\nu\lambda}, \quad (3)$$

$$[\mathbf{P}, \mathbf{P}']^{\lambda\mu\nu} = \mathbf{P}^{\rho\lambda} \partial_\rho \mathbf{P}'^{\mu\nu} + \mathbf{P}'^{\rho\lambda} \partial_\rho \mathbf{P}^{\mu\nu} + \mathbf{P}^{\rho\nu} \partial_\rho \mathbf{P}'^{\lambda\mu} + \mathbf{P}'^{\rho\nu} \partial_\rho \mathbf{P}^{\lambda\mu} + \mathbf{P}^{\rho\mu} \partial_\rho \mathbf{P}'^{\nu\lambda} + \mathbf{P}'^{\rho\mu} \partial_\rho \mathbf{P}^{\nu\lambda}, \quad (4)$$

where  $\partial_\rho = \frac{\partial}{\partial x_\rho}$  such that  $(x_1, \dots, x_m)$  is the coordinate of the manifold  $\mathbf{M}$ . The Poisson bracket corresponding to the Poisson bivector  $\mathbf{P}$  has the form

$$\{f, g\} = \mathbf{P}^{\mu\nu} \partial_\mu f \partial_\nu g. \quad (5)$$

The bracket (5) satisfies in the Jacobi identity i.e.  $\forall f, g, h \in C^\infty(\mathbf{M})$ ,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = [\mathbf{P}, \mathbf{P}]^{\lambda\mu\nu} \partial_\lambda f \partial_\mu g \partial_\nu h = 0 \quad (6)$$

if  $[\mathbf{P}, \mathbf{P}] = 0$  and vice versa. A manifold  $\mathbf{M}$  equipped with such compatible Poisson structures is called bi-Hamiltonian manifold. If a dynamical system on the manifold  $\mathbf{M}$  for which the number of functionally independent integrals of motion  $H_1, \dots, H_n$  are in bi-involution with respect to this compatible Poisson brackets,

$$\{H_i, H_j\} = \{H_i, H_j\}' = 0 \quad (7)$$

then the system is called bi-Hamiltonian system [9]. So to introduce the bi-Hamiltonian structure on the manifold  $\mathbf{M}$ , we must determine a pair of compatible and independent Poisson bivectors  $\mathbf{P}$  and  $\mathbf{P}'$ .

## 2.2 Compatible Poisson structures on Lie groups

Now we will try to simplify the relations (2), (3) and (4) when  $\mathbf{M}$  is a Lie group by using non-coordinate basis.

**Definition:** [12] In the coordinate basis,  $T_p \mathbf{M}$  spanned by  $\{e_\mu\} = \{\partial_\mu\}$  and  $T_p^* \mathbf{M}$  by  $\{dx^\mu\}$ , let us consider their linear combinations,

$$\mathbf{e}_i = e_i^\mu \partial_\mu, \quad \Theta^i = e^i_\mu dx^\mu, \quad \{e_i^\mu\} \in GL(m, R), \quad (8)$$

where  $\det e_i^\mu > 0$ . In other word,  $\{\mathbf{e}_i\}$  is the frame of basis vectors which is obtained by a  $GL(m, R)$ -rotation of the basis  $\{e_\mu\}$  and preserving the orientation. In the above  $e^i_\mu$  is inverse of  $e_i^\mu$  and we have

$$e^i_\mu e_i^\nu = \delta_\mu^\nu, \quad e^i_\mu e_j^\mu = \delta^i_j. \quad (9)$$

The bases  $\{\mathbf{e}_i\}$  and  $\{\Theta^i\}$  are called the non-coordinate bases. The coefficients  $e_i^\mu$  are called the vierbeins and we have

$$[\mathbf{e}_i, \mathbf{e}_j] = C_{ij}^k \mathbf{e}_k, \quad (10)$$

where  $C_{ij}^k$  is a function of coordinates of the manifold  $\mathbf{M}$ . When  $\mathbf{M}$  is a Lie group  $\mathbf{G}$ , these coefficients are the structure constants of the Lie algebra  $\mathfrak{g}$  of the Lie group  $\mathbf{G}$  and we have

$$C_{ij}^k = e^k_\nu (e_i^\mu \partial_\mu e_j^\nu - e_j^\mu \partial_\mu e_i^\nu). \quad (11)$$

Now we write the Poisson structure  $\mathbf{P}$  in terms of the non-coordinate basis as<sup>1</sup>

$$\mathbf{P}^{\mu\nu} = e_i^\mu e_j^\nu P^{ij}. \quad (12)$$

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<sup>1</sup>Here the indices  $\mu, \nu, \dots$  related to the coordinate of the group  $\mathbf{G}$  and the indices  $i, j, \dots$  related to the group parameter  $\{x_i\}$ . Furthermore, in the following we will consider  $\mathbf{G}$  as a phase space of the dynamical systems hence  $m = 2n$  must be even.

Note that in general  $P^{ij}$ 's are antisymmetric tensors and functions of the group parameters  $x_i$ 's. As a first case we consider  $P^{ij}$  and  $P'^{ij}$  as constants antisymmetric matrices

$$P = \begin{pmatrix} 0 & p_{12} & p_{13} & \dots & p_{1m} \\ -p_{12} & 0 & p_{23} & \dots & p_{2m} \\ -p_{13} & -p_{23} & 0 & \dots & p_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p_{1m} & -p_{2m} & -p_{3m} & \dots & 0 \end{pmatrix}, \quad P' = \begin{pmatrix} 0 & p'_{12} & p'_{13} & \dots & p'_{1m} \\ -p'_{12} & 0 & p'_{23} & \dots & p'_{2m} \\ -p'_{13} & -p'_{23} & 0 & \dots & p'_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p'_{1m} & -p'_{2m} & -p'_{3m} & \dots & 0 \end{pmatrix}, \quad (13)$$

where  $p_{ij}$  and  $p'_{ij}$  are real constants. Now using (1), (11) and (12) one can rewrite the relations (1) with (2), (3) and (4) as follows:

$$C_{ki}^s P^{kz} P^{i\gamma} + C_{kj}^\gamma P^{kz} P^{sj} + C_{ki}^z P^{k\gamma} P^{is} = 0, \quad (14)$$

$$C_{ki}^s P'^{kz} P'^{i\gamma} + C_{kj}^\gamma P'^{kz} P'^{sj} + C_{ki}^z P'^{k\gamma} P'^{is} = 0, \quad (15)$$

$$C_{ki}^s (P^{kz} P'^{i\gamma} + P'^{kz} P^{i\gamma}) + C_{kj}^\gamma (P^{kz} P'^{sj} + P'^{kz} P^{sj}) + C_{ki}^z (P^{k\gamma} P'^{is} + P'^{k\gamma} P^{is}) = 0. \quad (16)$$

then using the adjoint representation of the Lie algebra  $\mathfrak{g}$ , i.e.:  $(\mathcal{X}_i)_j^k = -C_{ij}^k$  and  $(\mathcal{Y}^k)_{ij} = -C_{ij}^k$ , one can rewrite the above relations in the following matrix forms:<sup>2</sup>

$$P \mathcal{X}_i P^{i\gamma} + P \mathcal{Y}^\gamma P + P^{i\gamma} \mathcal{X}_i^t P = 0, \quad (17)$$

$$P' \mathcal{X}_i P'^{i\gamma} + P' \mathcal{Y}^\gamma P' + P'^{i\gamma} \mathcal{X}_i^t P' = 0, \quad (18)$$

$$P \mathcal{X}_i P'^{i\gamma} + P' \mathcal{X}_i P^{i\gamma} + P \mathcal{Y}^\gamma P' + P' \mathcal{Y}^\gamma P + P^{i\gamma} \mathcal{X}_i^t P' + P'^{i\gamma} \mathcal{X}_i^t P = 0, \quad (19)$$

In this way having the structure constants of the Lie algebra  $\mathfrak{g}$ , one can solve the matrix equations (17) - (19) in order to obtain  $P$  and  $P'$ . Here we will consider four dimensional real Lie algebras [13]. For all of the symplectic<sup>3</sup> four dimensional real Lie algebras [15], one can see that all the solution of (17) - (19) are equivalent.

For this reason as a second case we consider  $P^{ij}$  as (13) but  $P'^{ij}$  as a linear functions of group parameters  $x_i$  of the Lie group  $\mathbf{G}$  as follows:

$$P' = \begin{pmatrix} 0 & p'_{12} + \sum_{i=1}^m a_{2i} x_i & p'_{13} + \sum_{i=1}^m a_{3i} x_i & \dots & p'_{1m} + \sum_{i=1}^m a_{mi} x_i \\ -p'_{12} - \sum_{i=1}^m a_{2i} x_i & 0 & p'_{23} + \sum_{i=1}^m b_{3i} x_i & \dots & p'_{2m} + \sum_{i=1}^m b_{mi} x_i \\ -p'_{13} - \sum_{i=1}^m a_{3i} x_i & -p'_{23} - \sum_{i=1}^m b_{3i} x_i & 0 & \dots & p'_{3m} + \sum_{i=1}^m c_{mi} x_i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p'_{1m} - \sum_{i=1}^m a_{mi} x_i & -p'_{2m} - \sum_{i=1}^m b_{mi} x_i & -p'_{3m} - \sum_{i=1}^m c_{mi} x_i & \dots & 0 \end{pmatrix}, \quad (20)$$

where  $p'_{ij}$  and  $a_{ij}$ 's are real constants; then relations (1) with (3) and (4) have the following matrix forms:

$$P' \mathcal{X}_i P'^{i\gamma} + P' \mathcal{Y}^\gamma P' + P'^{i\gamma} \mathcal{X}_i^t P' + (e^t P')^{k\gamma} \partial_k P' + A + B = 0, \quad (21)$$

$$P \mathcal{X}_i P'^{i\gamma} + P' \mathcal{X}_i P^{i\gamma} + P \mathcal{Y}^\gamma P' + P' \mathcal{Y}^\gamma P + P^{i\gamma} \mathcal{X}_i^t P' + P'^{i\gamma} \mathcal{X}_i^t P + (e^t P)^{k\gamma} \partial_k P' + A' + B' = 0, \quad (22)$$

where  $e^t$  is a transpose of the vierbeins  $e_\alpha^\mu$  and  $A, B, A'$  and  $B'$  have the following forms:

$$A = \begin{pmatrix} (e^t P')^{k1} \partial_k P'^{1\gamma} & (e^t P')^{k1} \partial_k P'^{2\gamma} & \dots & (e^t P')^{k1} \partial_k P'^{m\gamma} \\ (e^t P')^{k2} \partial_k P'^{1\gamma} & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ (e^t P')^{km} \partial_k P'^{1\gamma} & (e^t P')^{km} \partial_k P'^{2\gamma} & \dots & (e^t P')^{km} \partial_k P'^{m\gamma} \end{pmatrix} \quad (23)$$

<sup>2</sup>Here the upper index  $t$  represent the transpose of a matrix.

<sup>3</sup>Note that here we will consider symplectic four dimensional real Lie algebras [14] and not all of them [13], because we will construct integrable systems over those related Lie groups.

$$B = \begin{pmatrix} (e^t P')^{k1} \partial_k P'^{\gamma 1} & (e^t P')^{k2} \partial_k P'^{\gamma 1} & \dots & (e^t P')^{km} \partial_k P'^{\gamma 1} \\ (e^t P')^{k1} \partial_k P'^{\gamma 2} & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (e^t P')^{k1} \partial_k P'^{\gamma m} & (e^t P')^{k2} \partial_k P'^{\gamma m} & \dots & (e^t P')^{km} \partial_k P'^{\gamma m} \end{pmatrix} \quad (24)$$

$$A' = \begin{pmatrix} (e^t P)^{k1} \partial_k P'^{1\gamma} & (e^t P)^{k1} \partial_k P'^{2\gamma} & \dots & (e^t P)^{k1} \partial_k P'^{m\gamma} \\ (e^t P)^{k2} \partial_k P'^{1\gamma} & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (e^t P)^{km} \partial_k P'^{1\gamma} & (e^t P)^{km} \partial_k P'^{2\gamma} & \dots & (e^t P)^{km} \partial_k P'^{m\gamma} \end{pmatrix} \quad (25)$$

$$B' = \begin{pmatrix} (e^t P)^{k1} \partial_k P'^{\gamma 1} & (e^t P)^{k2} \partial_k P'^{\gamma 1} & \dots & (e^t P)^{km} \partial_k P'^{\gamma 1} \\ (e^t P)^{k1} \partial_k P'^{\gamma 2} & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (e^t P)^{k1} \partial_k P'^{\gamma m} & (e^t P)^{k2} \partial_k P'^{\gamma m} & \dots & (e^t P)^{km} \partial_k P'^{\gamma m} \end{pmatrix} \quad (26)$$

Now we will try to solve (17), (21) and (22) for four dimensional real Lie groups.

### 3 Some compatible Poisson structures on symplectic four dimensional real Lie groups

Having the structure constants of the Lie algebra  $\mathfrak{g}$ , we will solve matrix equations (17), (21) and (22) in order to obtain  $P$  (13) and  $P'$  (20). For self containing of the paper the list of symplectic four dimensional real Lie algebras [14] are brought in appendix A.

Let us consider an example; for Lie algebra  $A_{4,1}$  we have the following non zero commutators and the matrices  $\mathcal{X}_i$  and  $\mathcal{Y}^i$ :

$$[\mathbf{e}_2, \mathbf{e}_4] = \mathbf{e}_1, \quad [\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_2, \quad (27)$$

$$\mathcal{X}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{X}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{X}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \mathcal{X}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (28)$$

$$\mathcal{Y}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{Y}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{Y}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Y}^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (29)$$

also according to [16] for Lie algebra  $A_{4,1}$  the matrix  $e_\alpha^\mu$  has the following form:

$$(e_\alpha^\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_4 & 1 & 0 & 0 \\ x_4^2/2 & x_4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (30)$$

with substituting  $\mathcal{X}_i$ ,  $\mathcal{Y}^i$  and  $e_\alpha^\mu$  in (17), (21) and (22) one can obtain the compatible Poisson structures  $P$  and  $P'$  for Lie algebra  $A_{4,1}$ . One of the solution have the following forms:

$$P = \begin{pmatrix} 0 & p_{12} & 0 & p_{14} \\ * & 0 & p_{23} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}, \quad P' = \begin{pmatrix} 0 & p'_{12} - (a_{44} + p'_{24})x_2 + a_{23}x_3 + a_{24}x_4 & p'_{13} + a_{44}x_3 & a_{44}x_4 \\ * & 0 & 0 & p'_{24} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}. \quad (31)$$

Here  $p_{12}, p_{14}, p_{23}, p'_{12}, p'_{13}, p'_{24}, a_{23}, a_{44}$  and  $a_{24}$  arbitrary real constants. In this way we have obtained some compatible Poisson structures ( $P$  is constant and  $P'$  is linear function of the Lie group coordinates) on symplectic four dimensional real Lie algebras, the results are summarized in Table 1. Note that all parameters  $a_{ij}, p_{ij}$  and  $p'_{ij}$  are arbitrary real constants.

**Table 1:** Compatible Poisson structure on symplectic four dimensional real Lie algebras.

$\mathfrak{g}$	$P$	$P'$	Comments
$A_{4,1}$	$\begin{pmatrix} 0 & p_{12} & 0 & p_{14} \\ * & 0 & p_{23} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & p'_{12} - (a_{44} + p'_{24})x_2 + a_{23}x_3 + a_{24}x_4 & p'_{13} + a_{44}x_3 & a_{44}x_4 \\ * & 0 & 0 & p'_{24} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$p_{14}p_{23} \neq 0$
$A_{4,2}^{-1}$	$\begin{pmatrix} 0 & p_{12} & p_{13} & 0 \\ * & 0 & p_{23} & p_{24} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & p'_{12} + a_{23}x_3 + a_{24}x_4 & \frac{a_{53}p_{13}}{p_{23}}x_3 & 0 \\ * & 0 & a_{53}x_3 & p'_{24} + a_{64}x_4 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$p_{13}p_{24} \neq 0$
$A_{4,3}$	$\begin{pmatrix} 0 & p_{12} & 0 & p_{14} \\ * & 0 & p_{23} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & p'_{12} + a_{23}x_3 + a_{24}x_4 & a_{33}x_3 & a_{33}x_4 \\ * & 0 & a_{53}x_3 + \frac{a_{33}p_{23}}{p_{14}}x_4 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$p_{14}p_{23} \neq 0$
$A_{4,5}^{a,-a}$	$\begin{pmatrix} 0 & 0 & p_{13} & p_{14} \\ * & 0 & p_{23} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & p'_{13} + a_{32}x_2 + a_{34}x_4 & a_{44}x_4 \\ * & 0 & p'_{23} + a_{52}x_2 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$p_{14}p_{23} \neq 0$
$A_{4,5}^{-1,-1}$	$\begin{pmatrix} 0 & p_{12} & p_{13} & 0 \\ * & 0 & p_{23} & p_{24} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & p'_{12} + a_{23}x_3 + a_{24}x_4 & \frac{a_{53}p_{13}x_3}{p_{23}} & 0 \\ * & 0 & a_{53}x_3 & a_{64}x_4 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$p_{13}p_{24} \neq 0$
$A_{4,6}^{a,0}$	$\begin{pmatrix} 0 & 0 & 0 & p_{14} \\ * & 0 & p_{23} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & a_{41}x_1 + a_{44}x_4 \\ * & 0 & a_{52}x_2 + a_{53}x_3 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$p_{14}p_{23} \neq 0$
$A_{4,9}^0$	$\begin{pmatrix} 0 & 0 & p_{13} & p_{23} \\ * & 0 & p_{23} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & p'_{13} + a_{52}x_1 + a_{32}x_2 + a_{54}x_3 + a_{34}x_4 & p'_{23} + a_{52}x_2 + a_{54}x_4 \\ * & 0 & p'_{23} + a_{52}x_2 + a_{54}x_4 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$p_{23} \neq 0$
$A_{4,12}$	$\begin{pmatrix} 0 & 0 & 0 & p_{14} \\ * & 0 & -p_{14} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -a_{54}x_3 + a_{64}x_4 & -a_{64}x_3 - a_{54}x_4 \\ * & 0 & a_{64}x_3 + a_{54}x_4 & -a_{54}x_3 + a_{64}x_4 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$p_{14} \neq 0$
$II \oplus R$	$\begin{pmatrix} 0 & p_{12} & p_{13} & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & p_{34} \\ * & * & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a_{44}x_2 & p'_{13} + a_{32}x_2 - (a_{44} + p'_{23})x_3 & a_{44}x_4 \\ * & 0 & p'_{23} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$p_{12}p_{34} \neq 0$
$III \oplus R$	$\begin{pmatrix} 0 & p_{13} & p_{13} & 0 \\ * & 0 & 0 & p_{24} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a_{33}x_3 & a_{33}x_3 & 0 \\ * & 0 & 0 & a_{64}x_4 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$p_{13}p_{24} \neq 0$
$VI_0 \oplus R$	$\begin{pmatrix} 0 & p_{12} & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & p_{34} \\ * & * & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & p'_{12} + a_{22}x_1 + a_{22}x_2 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & p'_{34} + a_{73}x_3 + a_{74}x_4 \\ * & * & * & 0 \end{pmatrix}$	$p_{12}p_{34} \neq 0$
$VII_0 \oplus R$	$\begin{pmatrix} 0 & p_{12} & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & p_{34} \\ * & * & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a_{21}x_1 + a_{22}x_2 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & a_{73}x_3 + a_{74}x_4 \\ * & * & * & 0 \end{pmatrix}$	$p_{12}p_{34} \neq 0$
$A_2 \oplus A_2$	$\begin{pmatrix} 0 & p_{12} & 0 & 0 \\ * & 0 & 0 & p_{24} \\ * & * & 0 & p_{34} \\ * & * & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a_{21}x_1 & 0 & 0 \\ * & 0 & 0 & p'_{24} + a_{61}x_1 + a_{63}x_3 \\ * & * & 0 & a_{73}x_3 \\ * & * & * & 0 \end{pmatrix}$	$p_{12}p_{34} \neq 0$

## 4 Integrable bi-Hamiltonian systems on symplectic four dimensional real Lie groups

In this section, we will construct the integrable bi-Hamiltonian systems with four dimensional real Lie groups as phase space. For this propose, in the previous section we have considered those four dimensional real Lie groups such that they have symplectic structures [15], [14]. Here, we will construct the models on those Lie groups separately as follows.

An important class of bi-Hamiltonian manifold occurs when one of the compatible Poisson structures is invertible i.e., the Poisson bracket  $\{.,.\}$  associated whit  $\mathbf{P}$  is invertible. Then one can define a linear map  $\mathbf{N} : TM \longrightarrow TM$

acting on the tangent bundle by [9]

$$\mathbf{N} = \mathbf{P}' \mathbf{P}^{-1}. \quad (32)$$

**Theorem (Magri-Morosi)**[8], [2]: *A remarkable consequence of the compatibility of  $\mathbf{P}$  and  $\mathbf{P}'$  is that the torsion of Nijenhuis tensor  $\mathbf{N}$*

$$T_{\mathbf{N}}(X, Y) = [\mathbf{N}X, \mathbf{N}Y] - \mathbf{N}[\mathbf{N}X, Y] - \mathbf{N}[X, \mathbf{N}Y] + \mathbf{N}^2[X, Y] \quad (33)$$

*identically vanishes ; where  $X$  and  $Y$  are arbitrary vector fields and the bracket  $[X, Y]$  denotes the Lie bracket (commutator). One of the main properties of  $\mathbf{N}$  is that the normalized traces of the powers of  $\mathbf{N}$  are integrals of motion*

$$H_k = \frac{1}{2k} \text{Tr} \mathbf{N}^k, \quad (34)$$

*and satisfying Lenard-Magri recurrent relations [9].*

$$\mathbf{P}' dH_i = \mathbf{P} dH_{i+1}, \quad (35)$$

In the table 1,  $P$  and  $P'$  on the Lie algebras were presented. Now with substituting  $P$  and  $P'$  in (12) and using the related vierbeins [16] the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on Lie groups are obtained, then using (32) and (34) one can find the Hamiltonian and integrals of motions of bi-Hamiltonian systems. In the following we will perform this work separately for symplectic four dimensional real Lie groups. In this way we obtain new bi-Hamiltonian systems over four dimensional real Lie groups as phase spaces.

#### Lie group $\mathbf{A}_{4,1}$ :

With substituting  $P$  and  $P'$  in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $\mathbf{A}_{4,1}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & p_{12} + \frac{p_{23}x_4^2}{2} & p_{23}x_4 & p_{14} \\ * & 0 & p_{23} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}, \quad (36)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & p'_{12} - (a_{44} - p'_{24})x_2 + a_{23}x_3 + a_{24}x_4 + p'_{13}x_4 + a_{44}x_3x_4 & p'_{13} + a_{44}x_3 & (p'_{24} + a_{44})x_4 \\ * & 0 & 0 & p'_{24} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}. \quad (37)$$

Now by means of (32) and (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{a_{44}x_4}{p_{14}}, \quad H_2 = \frac{2p_{14}p'_{24}(p'_{13} + a_{44}x_3) + a_{44}^2p_{23}x_4^2}{2p_{14}^2p_{23}}. \quad (38)$$

#### Lie group $\mathbf{A}_{4,2}^{-1}$ :

Substituting  $P$  and  $P'$  in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $\mathbf{A}_{4,2}^{-1}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & p_{12} + p_{13}x_4 & p_{13} & 0 \\ * & 0 & e^{2x_4}p_{23} & e^{x_4}p_{24} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}, \quad (39)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & p'_{12} + a_{23}x_3 + a_{24}x_4 + \frac{a_{53}p_{13}x_3x_4}{p_{23}} & \frac{a_{53}p_{13}x_3}{p_{23}} & 0 \\ * & 0 & a_{53}e^{2x_4}x_3 & e^{x_4}(p'_{24} + a_{64}x_4) \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}. \quad (40)$$

By means of (32) and (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{p_{23}p'_{24} + a_{53}p_{24}x_3 + a_{64}p_{23}x_4}{p_{23}p_{24}}, \quad H_2 = 1/4 \left( \frac{2a_{53}^2x_3^2}{p_{23}^2} + \frac{2(p'_{24} + a_{64}x_4)^2}{p_{24}^2} \right). \quad (41)$$

**Lie group  $A_{4,3}$ :**

With substituting  $P$  and  $P'$  in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $A_{4,3}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & p_{12}e^{x_4} & 0 & p_{14}e^{x_4} \\ * & 0 & p_{23} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}, \quad (42)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & e^{x_4}(p'_{12} + a_{23}x_3 + a_{24}x_4 + a_{33}x_3x_4) & a_{33}e^{x_4}x_3 & a_{33}e^{x_4}x_4 \\ * & 0 & a_{53}x_3 + \frac{a_{33}p_{23}x_4}{p_{14}} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}. \quad (43)$$

Now by means of (32) and (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{a_{53}x_3}{p_{23}} + \frac{2a_{33}x_4}{p_{14}}, \quad H_2 = 1/2\left(\frac{a_{33}^2x_4^2}{p_{14}^2} + \left(\frac{a_{53}x_3}{p_{23}} + \frac{a_{33}x_4}{p_{14}}\right)^2\right). \quad (44)$$

**Lie group  $A_{4,5}^{a,-a}$ :**

With substituting  $P$  and  $P'$  in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $A_{4,5}^{a,-a}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & p_{13}e^{(1-a)x_4} & p_{14}e^{x_4} \\ * & 0 & p_{23} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}, \quad (45)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & 0 & e^{(1-a)x_4}(p'_{13} + a_{32}x_2 + a_{34}x_4) & a_{44}e^{x_4}x_4 \\ * & 0 & p'_{23} + a_{52}x_2 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}. \quad (46)$$

Now by means of (32) and (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{p'_{23} + a_{52}x_2}{p_{23}} + \frac{a_{44}x_4}{p_{14}}, \quad H_2 = 1/2\left(\frac{(p'_{23} + a_{52}x_2)^2}{p_{23}^2} + \frac{a_{44}^2x_4^2}{p_{14}^2}\right). \quad (47)$$

**Lie group  $A_{4,5}^{-1,-1}$ :**

With substituting  $P$  and  $P'$  in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $A_{4,5}^{-1,-1}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & p_{12} & p_{13} & 0 \\ * & 0 & p_{23}e^{-2x_4} & p_{24}e^{-x_4} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}, \quad (48)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & p'_{12} + a_{23}x_3 + a_{24}x_4 & \frac{a_{53}p_{13}x_3}{p_{23}} & 0 \\ * & 0 & a_{53}e^{-2x_4}x_3 & a_{64}e^{-x_4}x_4 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}. \quad (49)$$

Now by means of (32) and (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{a_{53}x_3}{p_{23}} + \frac{a_{64}x_4}{p_{24}}, \quad H_2 = 1/2\left(\left(\frac{a_{53}x_3}{p_{23}}\right)^2 + \left(\frac{a_{64}x_4}{p_{24}}\right)^2\right). \quad (50)$$

**Lie group  $A_{4,6}^{a,0}$ :**

With substituting  $P$  and  $P'$  in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $A_{4,6}^{a,0}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & p_{14}e^{ax_4} \\ * & 0 & p_{23} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}, \quad (51)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & 0 & 0 & e^{ax_4}(a_{41}x_1 + a_{44}x_4) \\ * & 0 & a_{52}x_2 + a_{53}x_3 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}. \quad (52)$$

Now by means of (32) and (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{a_{41}x_1 + a_{44}x_4}{p_{14}} + \frac{a_{52}x_2 + a_{53}x_3}{p_{23}}, \quad H_2 = 1/2((\frac{a_{41}x_1 + a_{44}x_4}{p_{14}})^2 + (\frac{a_{52}x_2 + a_{53}x_3}{p_{23}})^2). \quad (53)$$

**Lie group  $\mathbf{A}_{4,12}$ :**

With substituting  $P$  and  $P'$  in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $\mathbf{A}_{4,12}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & -p_{14}e^{x_3}\sin(x_4) & p_{14}e^{x_3}\cos(x_4) \\ * & 0 & -p_{14}e^{x_3}\cos(x_4) & -p_{14}e^{x_3}\sin(x_4) \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}, \quad (54)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & 0 & e^{x_3}(-\alpha\cos(x_4) + \beta\sin(x_4)) & -e^{x_3}(\beta\cos(x_4) + \alpha\sin(x_4)) \\ * & 0 & e^{x_3}(\beta\cos(x_4) + \alpha\sin(x_4)) & e^{x_3}(-\alpha\cos(x_4) + \beta\sin(x_4)) \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}, \quad (55)$$

where  $\alpha = a_{54}x_3 - a_{64}x_4$  and  $\beta = a_{64}x_3 + a_{54}x_4$ . Now by means of (32) and (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{-2a_{64}x_3 - 2a_{54}x_4}{p_{14}}, \quad H_2 = \frac{(a_{64}^2 - a_{54}^2)x_3^2 + 4a_{54}a_{64}x_3x_4 + (a_{54}^2 - a_{64}^2)x_4^2}{p_{14}^2}. \quad (56)$$

**Lie group  $\mathbf{A}_2 \oplus \mathbf{A}_2$ :**

With substituting  $P$  and  $P'$  in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $\mathbf{A}_2 \oplus \mathbf{A}_2$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & p_{12} & 0 & 0 \\ * & 0 & 0 & p_{24} \\ * & * & 0 & p_{34} \\ * & * & * & 0 \end{pmatrix}, \quad (57)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & a_{21}x_1 & 0 & 0 \\ * & 0 & 0 & p'_{24} + a_{61}x_1 + a_{63}x_3 \\ * & * & 0 & a_{73}x_3 \\ * & * & * & 0 \end{pmatrix}. \quad (58)$$

Now by means of (32) and (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{a_{21}x_1}{p_{12}} + \frac{a_{73}x_3}{p_{34}}, \quad H_2 = \frac{a_{21}^2x_1^2}{2p_{12}^2} + \frac{a_{73}^2x_3^2}{2p_{34}^2}. \quad (59)$$

**Lie group  $\mathbf{VII}_0 \oplus \mathbf{R}$ :**

With substituting  $P$  and  $P'$  in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $\mathbf{VII}_0 \oplus \mathbf{R}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & p_{12} & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & p_{34} \\ * & * & * & 0 \end{pmatrix}, \quad (60)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & a_{21}x_1 + a_{22}x_2 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & a_{73}x_3 + a_{74}x_4 \\ * & * & * & 0 \end{pmatrix}. \quad (61)$$



Now by means of (32) and (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{a_{21}x_1 + a_{22}x_2}{p_{12}} + \frac{a_{73}x_3 + a_{74}x_4}{p_{34}}, \quad H_2 = 1/2((\frac{a_{21}x_1 + a_{22}x_2}{p_{12}})^2 + (\frac{a_{73}x_3 + a_{74}x_4}{p_{34}})^2). \quad (62)$$

**Lie group  $\mathbf{VI}_0 \oplus \mathbf{R}$ :**

With substituting  $P$  and  $P'$  in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $\mathbf{VI}_0 \oplus \mathbf{R}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & p_{12} & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & p_{34} \\ * & * & * & 0 \end{pmatrix}, \quad (63)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & p'_{12} + a_{22}(x_1 + x_2) & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & p'_{34} + a_{73}x_3 + a_{74}x_4 \\ * & * & * & 0 \end{pmatrix}. \quad (64)$$

Now by means of (32) and (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{p'_{12} + a_{22}x_1 + a_{22}x_2}{p_{12}} + \frac{p'_{34} + a_{73}x_3 + a_{74}x_4}{p_{34}}, \quad H_2 = \frac{1}{2}((\frac{p'_{12} + a_{22}x_1 + a_{22}x_2}{p_{12}})^2 + (\frac{p'_{34} + a_{73}x_3 + a_{74}x_4}{p_{34}})^2). \quad (65)$$

**Lie group  $\mathbf{III} \oplus \mathbf{R}$ :**

With substituting  $P$  and  $P'$  in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $\mathbf{III} \oplus \mathbf{R}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & p_{13} & p_{13} & 0 \\ * & 0 & 0 & p_{24} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}, \quad (66)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & a_{33}x_3 & a_{33}x_3 & 0 \\ * & 0 & 0 & a_{64}x_4 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}. \quad (67)$$

Now by means of (32) and (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{a_{33}x_3}{p_{13}} + \frac{a_{64}x_4}{p_{24}}, \quad H_2 = \frac{1}{2}((\frac{a_{33}x_3}{p_{13}})^2 + (\frac{a_{64}x_4}{p_{24}})^2). \quad (68)$$

**Lie group  $\mathbf{II} \oplus \mathbf{R}$ :**

With substituting  $P$  and  $P'$  in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $\mathbf{II} \oplus \mathbf{R}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & p_{12} & p_{13} & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & p_{34} \\ * & * & * & 0 \end{pmatrix}, \quad (69)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & a_{44}x_2 & p'_{13} + a_{32}x_2 - a_{44}x_3 & a_{44}x_4 \\ * & 0 & p'_{23} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}. \quad (70)$$

Now by means of (32) and (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{a_{44}x_2}{p_{12}}, \quad H_2 = \frac{a_{44}(a_{44}p_{34}x_2^2 - 2p_{12}p'_{23}x_4)}{2p_{12}^2p_{34}}. \quad (71)$$

## 5 Some compatible Poisson structures on symplectic nilpotent six dimensional real Lie algebras

In this section, we will solve matrix equations (17), (21) and (22) for symplectic nilpotent six dimensional real Lie groups in order to obtain  $P$  and  $P'$ . For self containing of the paper the list of symplectic six dimensional real Lie algebras [17] are brought in appendix B. Note that for calculating  $P$  and  $P'$  from (21) and (22) we must first calculate vielbeins  $e_\alpha^\mu$  for nilpotent 6-dimensional real Lie groups. To this end, we use the following relation:

$$g^{-1}dg = e_\alpha^\mu X_\mu dx^\alpha \quad g \in \mathbf{G} \quad (72)$$

With the following parameterizations for the real 6-dimensional Lie groups  $\mathbf{G}$ :

$$g = e^{x_1 X_1} e^{x_2 X_2} e^{x_3 X_3} e^{x_4 X_4} e^{x_5 X_5} e^{x_6 X_6}, \quad (73)$$

where  $X_i$  and  $x_i$  are generators and coordinates of Lie group, respectively. Then, for left invariant Lie algebra valued one forms, we have:

$$\begin{aligned} g^{-1}dg &= dx_1 e^{-x_6 X_6} e^{-x_5 X_5} e^{-x_4 X_4} e^{-x_3 X_3} (e^{-x_2 X_2} X_1 e^{x_2 X_2}) e^{x_3 X_3} e^{x_4 X_4} e^{x_5 X_5} e^{x_6 X_6} \\ &\quad + dx_2 e^{-x_6 X_6} e^{-x_5 X_5} e^{-x_4 X_4} (e^{-x_3 X_3} X_2 e^{x_3 X_3}) e^{x_4 X_4} e^{x_5 X_5} e^{x_6 X_6} \\ &\quad + dx_3 e^{-x_6 X_6} e^{-x_5 X_5} (e^{-x_4 X_4} X_3 e^{x_4 X_4}) e^{x_5 X_5} e^{x_6 X_6} \\ &\quad + dx_4 e^{-x_6 X_6} (e^{-x_5 X_5} X_4 e^{x_5 X_5}) e^{x_6 X_6} + dx_5 e^{-x_6 X_6} X_5 e^{x_6 X_6} + dx_6 X_6 \end{aligned} \quad (74)$$

such that, for this calculation one can use the following relation [18]

$$(e^{-x_i X_i} X_j e^{x_i X_i}) = (e^{x_i X_i})^k X_k, \quad (75)$$

in which we have a summation over the index  $k$  on the right hand side but there is not summation on the index  $i$ . In this way one can calculate all left invariant one forms and vierbeins. The list of symplectic nilpotent six dimensional real Lie algebras and vierbein matrices are given in appendix B.

Let us consider an example for calculating of  $P$  and  $P'$ ; for Lie algebra  $A_{6,1}$  we have the following non zero commutators and the matrices  $\mathcal{X}_i$  and  $\mathcal{Y}^i$ :

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_4, \quad [\mathbf{e}_1, \mathbf{e}_5] = \mathbf{e}_6, \quad (76)$$

$$\begin{aligned} \mathcal{X}_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{X}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{X}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \mathcal{X}_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{X}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{X}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \mathcal{Y}^1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Y}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Y}^3 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \mathcal{Y}^4 &= \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Y}^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Y}^6 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (77)$$

for Lie algebra  $A_{6,1}$  the vierbein matrix  $e_\alpha^\mu$  has the following form:

$$(e_\alpha^\mu) = \begin{pmatrix} 1 & 0 & x_2 & x_3 & 0 & x_5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (78)$$

Now with substituting  $\mathcal{X}_i, \mathcal{Y}^i$  and  $e_\alpha^\mu$  in (17), (21) and (22) one can obtain the compatible Poisson structures  $P$  and  $P'$  for Lie algebra  $A_{6,1}$ . One of the solution have the following forms:

$$P = \begin{pmatrix} 0 & 0 & 0 & p_{14} & 0 & 0 \\ * & 0 & p_{23} & 0 & 0 & 0 \\ * & * & 0 & p_{34} & 0 & p_{36} \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & p_{56} \\ * & * & * & * & * & 0 \end{pmatrix}, P' = \begin{pmatrix} 0 & 0 & 0 & \frac{c_{44}p_{14}x_4}{p_{34}} & 0 & 0 \\ * & 0 & b_{32}x_2 & 0 & 0 & 0 \\ * & * & 0 & c_{44}x_4 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & e_{65}x_5 \\ * & * & * & * & * & 0 \end{pmatrix}. \quad (79)$$

In this way we have obtained some compatible Poisson structures ( $P$  is constant and  $P'$  as linear function of Lie group coordinates) on symplectic nilpotent six dimensional real Lie algebras, the results are summarized in Table 2.

**Table 2:** Compatible Poisson structures on nilpotent 6-dimensional real Lie algebra.

g		Non-zero Poisson structure relations	Comments
$A_{6,1}$	$P$	$\{x_1, x_4\} = p_{14}$ , $\{x_2, x_3\} = p_{23}$ , $\{x_3, x_4\} = p_{34}$ , $\{x_3, x_5\} = p_{36}$ , $\{x_5, x_6\} = p_{56}$	$p_{14}p_{26} \neq 0$
	$P'$	$\{x_1, x_4\} = \frac{c_{44}p_{14}x_4}{p_{34}}$ , $\{x_2, x_3\} = b_{32}x_2$ , $\{x_3, x_4\} = c_{44}x_4$ , $\{x_5, x_6\} = e_{65}x_5$	
$A_{6,7}$	$P$	$\{x_1, x_5\} = p_{15}$ , $\{x_2, x_6\} = p_{26}$ , $\{x_3, x_4\} = p_{34}$ , $\{x_4, x_5\} = p_{45}$	$p_{15}p_{26}p_{34} \neq 0$
	$P'$	$\{x_1, x_5\} = \frac{a_{51}x_1}{p_{34}}$ , $\{x_2, x_6\} = b_{62}x_2 + b_{66}x_6$ , $\{x_3, x_4\} = c_{43}x_3$	
$A_{6,9}$	$P$	$\{x_1, x_4\} = p_{14}$ , $\{x_2, x_6\} = -p_{35}$ , $\{x_3, x_5\} = p_{35}$ , $\{x_3, x_6\} = p_{36}$ , $\{x_4, x_6\} = p_{46}$ , $\{x_5, x_6\} = p_{56}$	$p_{14}p_{35} \neq 0$
	$P'$	$\{x_1, x_4\} = a_{41}x_1 + a_{44}x_4 + \frac{a_{44}p_{46}x_2}{p_{35}}$ , $\{x_2, x_6\} = b_{62}x_2 + b_{66}x_6$ , $\{x_2, x_6\} = -\frac{d_{62}p_{35}x_2}{p_{46}}$ , $\{x_3, x_5\} = \frac{d_{62}p_{35}x_2}{p_{46}}$ , $\{x_3, x_6\} = c_{62}x_2 + c_{65}x_5$ , $\{x_4, x_6\} = d_{62}x_2$ , $\{x_5, x_6\} = e_{62}x_2 - \frac{d_{62}p_{35}x_5}{p_{46}}$	
$A_{6,24}$	$P$	$\{x_1, x_3\} = p_{13}$ , $\{x_2, x_5\} = p_{25}$ , $\{x_4, x_6\} = p_{46}$	$p_{13}p_{25}p_{46} \neq 0$
	$P'$	$\{x_1, x_3\} = a_{31}x_1 + a_{33}x_3$ , $\{x_2, x_5\} = b_{52}x_2 + b_{55}x_5$ , $\{x_4, x_6\} = d_{64}x_4 + d_{66}x_6$	
$A_{6,25}$	$P$	$\{x_1, x_3\} = p_{13}$ , $\{x_2, x_4\} = p_{24}$ , $\{x_3, x_6\} = p_{36}$ , $\{x_5, x_6\} = p_{56}$	$p_{13}p_{24}p_{56} \neq 0$
	$P'$	$\{x_1, x_3\} = a_{31}x_1$ , $\{x_2, x_4\} = b_{42}x_2 + b_{44}x_4$ , $\{x_5, x_6\} = e_{65}x_5$	
$A_{6,26}$	$P$	$\{x_1, x_4\} = p_{14}$ , $\{x_2, x_3\} = p_{23}$ , $\{x_3, x_4\} = \frac{c_{44}p_{14}}{a_{44}}$ , $\{x_3, x_5\} = p_{35}$ , $\{x_3, x_6\} = \frac{d_{64}p_{35}}{d_{54}}$ , $\{x_4, x_5\} = \frac{d_{54}p_{14}}{a_{44}}$ , $\{x_4, x_6\} = \frac{d_{64}p_{14}}{a_{44}}$ , $\{x_5, x_6\} = p_{56}$	$p_{13}p_{24}p_{56} \neq 0$
	$P'$	$\{x_1, x_4\} = a_{44}x_4$ , $\{x_2, x_3\} = b_{32}x_2$ , $\{x_3, x_4\} = c_{44}x_4$ , $\{x_4, x_5\} = d_{54}x_4$ , $\{x_4, x_6\} = d_{64}x_4$ , $\{x_5, x_6\} = e_{65}x_5 - \frac{d_{54}e_{65}x_6}{d_{64}}$	
$A_{6,27}$	$P$	$\{x_1, x_2\} = p_{12}$ , $\{x_1, x_3\} = p_{13}$ , $\{x_2, x_4\} = p_{24}$ , $\{x_5, x_6\} = p_{56}$	$p_{13}p_{24}p_{56} \neq 0$
	$P'$	$\{x_1, x_3\} = a_{33}x_3$ , $\{x_2, x_4\} = b_{44}x_4$ , $\{x_5, x_6\} = e_{65}x_5 + e_{66}x_6$	
$A_{6,32}$	$P$	$\{x_1, x_2\} = p_{12}$ , $\{x_1, x_4\} = -p_{23}$ , $\{x_1, x_6\} = \frac{a_{63}p_{23} + a_{25}p_{56}}{b_{33}}$ , $\{x_2, x_3\} = p_{23}$ , $\{x_5, x_6\} = p_{56}$	$p_{23}p_{56} \neq 0$
	$P'$	$\{x_1, x_2\} = -b_{33}x_1 - b_{34}x_2 + a_{23}x_3 + a_{24}x_4 + a_{25}x_5$ , $\{x_1, x_4\} = -b_{33}x_3 - b_{34}x_4$ , $\{x_1, x_6\} = a_{63}x_3 + \frac{a_{63}(b_{33}b_{34} + a_{25}e_{66})x_4}{b_{33}^2} + \frac{a_{25}e_{65}x_5 + a_{25}e_{66}x_6}{b_{33}}$ , $\{x_2, x_3\} = b_{33}x_3 + b_{34}x_4$ , $\{x_5, x_6\} = \frac{a_{63}e_{66}x_4}{b_{33}} + e_{65}x_5 + e_{66}x_6$	

## 6 Integrable bi-Hamiltonian systems on symplectic nilpotent six dimensional real Lie groups

In this section, we construct the integrable bi-Hamiltonian systems with symplectic nilpotent six dimensional real Lie groups as phase space. In the table 2,  $P$  and  $P'$  on the Lie algebras are presented. Now with substituting  $P$  and  $P'$  in (12) and using the related vierbeins as in appendix B the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on Lie groups are obtained, then using (32) and (34) one can find the Hamiltonian and integrals of motions of bi-Hamiltonian systems. In the following we perform this work separately for symplectic nilpotent six dimensional real Lie groups. In this way we obtain new bi-Hamiltonian systems over nilpotent six dimensional real Lie groups as phase spaces.

### Lie group $A_{6,1}$ :

With substituting  $P, P'$  and vierbein matrix in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and

$\mathbf{P}'$  on the Lie group  $\mathbf{A}_{6,1}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & p_{14} & 0 & 0 \\ * & 0 & p_{23} & 0 & 0 & 0 \\ * & * & 0 & p_{34} + p_{14}x_2 & 0 & p_{36} \\ * & * & * & 0 & 0 & -p_{14}x_5 \\ * & * & * & * & 0 & p_{56} \\ * & * & * & * & * & 0 \end{pmatrix}, \quad (80)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & 0 & 0 & \frac{c_{44}p_{14}x_4}{p_{34}} & 0 & 0 \\ * & 0 & b_{32}x_2 & 0 & 0 & 0 \\ * & * & 0 & c_{44}x_4 + \frac{c_{44}p_{14}x_2x_4}{p_{34}} & 0 & p_{36} \\ * & * & * & 0 & 0 & -\frac{c_{44}p_{14}x_4x_5}{p_{34}} \\ * & * & * & * & 0 & e_{65}x_5 \\ * & * & * & * & * & 0 \end{pmatrix}. \quad (81)$$

Now by means of (32) and (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{b_{32}x_2}{p_{23}} + \frac{c_{44}x_4}{p_{34}} + \frac{e_{65}x_5}{p_{56}}, \quad H_2 = 1/2((\frac{b_{32}x_2}{p_{23}})^2 + (\frac{c_{44}x_4}{p_{34}})^2 + (\frac{e_{65}x_5}{p_{56}})^2),$$

$$H_3 = 1/2((\frac{b_{32}x_2}{p_{23}})^3 + (\frac{c_{44}x_4}{p_{34}})^3 + (\frac{e_{65}x_5}{p_{56}})^3). \quad (82)$$

#### Lie group $\mathbf{A}_{6,7}$ :

With substituting  $P$ ,  $P'$  and vierbein matrix in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $\mathbf{A}_{6,7}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & p_{15} & 0 \\ * & 0 & 0 & 0 & 0 & p_{26} \\ * & * & 0 & p_{34} & 0 & 0 \\ * & * & * & 0 & p_{45} + p_{15}x_3 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{pmatrix}, \quad (83)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & 0 & 0 & 0 & a_{51}x_1 & 0 \\ * & 0 & 0 & 0 & 0 & b_{62}x_2 + b_{66}x_6 \\ * & * & 0 & c_{43}x_3 & 0 & 0 \\ * & * & * & 0 & a_{51}x_1x_3 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{pmatrix}. \quad (84)$$

Now by means of (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{a_{51}x_1}{p_{15}} + \frac{b_{62}x_2}{p_{26}} + \frac{c_{43}x_3}{p_{34}} + \frac{b_{66}x_6}{p_{26}}, \quad H_2 = 1/2((\frac{a_{51}x_1}{p_{15}})^2 + (\frac{c_{43}x_3}{p_{34}})^2 + (\frac{b_{62}x_2 + b_{66}x_6}{p_{26}})^2),$$

$$H_3 = 1/2((\frac{a_{51}x_1}{p_{15}})^3 + (\frac{c_{43}x_3}{p_{34}})^3 + (\frac{b_{62}x_2 + b_{66}x_6}{p_{26}})^3). \quad (85)$$

#### Lie group $\mathbf{A}_{6,9}$ :

With substituting  $P$ ,  $P'$  and vierbein matrix in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $\mathbf{A}_{6,9}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & p_{14} & 0 & 0 \\ * & 0 & 0 & 0 & 0 & -p_{35} \\ * & * & 0 & p_{14}x_2 & p_{35} & p_{36} \\ * & * & * & 0 & 0 & p_{46} + 1/2p_{14}x_2^2 - p_{14}x_5 \\ * & * & * & * & 0 & p_{56} \\ * & * & * & * & * & 0 \end{pmatrix}, \quad (86)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & 0 & 0 & a_{41}x_1 + \frac{a_{44}p_{46}x_2}{p_{35}} + a_{44}x_4 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & -\frac{d_{62}p_{35}x_2}{p_{46}} \\ * & * & 0 & a_{41}x_1x_2 + \frac{a_{44}p_{46}x_2^2}{p_{35}} + a_{44}x_2x_4 & \frac{c_{62}x_2 + c_{65}x_5}{2p_{35}} & \frac{2d_{62}p_{35}x_2 + (a_{41}p_{35}x_1 + a_{44}p_{46}x_2 + a_{44}p_{35}x_4)(x_2^2 - 2x_5)}{2p_{35}} \\ * & * & * & 0 & 0 & \frac{c_{62}x_2 - \frac{d_{62}p_{35}x_5}{p_{46}}}{0} \\ * & * & * & * & 0 & * \\ * & * & * & * & * & 0 \end{pmatrix}. \quad (87)$$

Now by means of (34), the integrals of motion could be found for this Lie group as follows:

$$\begin{aligned} H_1 &= \frac{a_{41}p_{35}p_{46}x_1 + 2d_{62}p_{14}p_{35}x_2 + a_{44}p_{46}(p_{46}x_2 + p_{35}x_4)}{p_{14}p_{35}p_{46}}, \\ H_2 &= 1/4((\frac{4d_{62}x_2}{p_{46}})^2 + (\frac{a_{41}p_{35}x_1 + a_{44}p_{46}x_2 + a_{44}p_{35}x_4}{p_{14}p_{35}})^2 + (\frac{a_{41}x_1 + a_{44}(\frac{p_{46}x_2}{p_{35}} + x_4)}{p_{14}})^2), \\ H_3 &= 1/6((\frac{4d_{62}x_2}{p_{46}})^3 + (\frac{a_{41}p_{35}x_1 + a_{44}p_{46}x_2 + a_{44}p_{35}x_4}{p_{14}p_{35}})^3 + (\frac{a_{41}x_1 + a_{44}(\frac{p_{46}x_2}{p_{35}} + x_4)}{p_{14}})^3). \end{aligned} \quad (88)$$

#### Lie group $\mathbf{A}_{6,24}$ :

With substituting  $P$ ,  $P'$  and vierbein matrix in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $\mathbf{A}_{6,24}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & p_{13} & 0 & 0 & 0 \\ * & 0 & 0 & 0 & p_{25} & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & p_{46} \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{pmatrix}, \quad (89)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & 0 & a_{31}x_1 + a_{33}x_3 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & b_{52}x_2 + b_{55}x_5 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & d_{64}x_4 + d_{66}x_6 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{pmatrix}. \quad (90)$$

Now by means of (34), the integrals of motion could be found for this Lie group as follows:

$$\begin{aligned} H_1 &= \frac{a_{31}x_1 + a_{33}x_3}{p_{13}} + \frac{b_{52}x_2 + b_{55}x_5}{p_{25}} + \frac{d_{64}x_4 + d_{66}x_6}{p_{45}}, \quad H_2 = 1/2((\frac{a_{31}x_1 + a_{33}x_3}{p_{13}})^2 + (\frac{b_{52}x_2 + b_{55}x_5}{p_{25}})^2 + (\frac{d_{64}x_4 + d_{66}x_6}{p_{45}})^2), \\ H_3 &= 1/3((\frac{a_{31}x_1 + a_{33}x_3}{p_{13}})^3 + (\frac{b_{52}x_2 + b_{55}x_5}{p_{25}})^3 + (\frac{d_{64}x_4 + d_{66}x_6}{p_{45}})^3). \end{aligned} \quad (91)$$

#### Lie group $\mathbf{A}_{6,25}$ :

With substituting  $P$ ,  $P'$  and vierbein matrix in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $\mathbf{A}_{6,25}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & p_{13} & 0 & 0 & 0 \\ * & 0 & 0 & p_{24} & 0 & p_{24}x_5 \\ * & * & 0 & 0 & 0 & p_{36} \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & p_{56} \\ * & * & * & * & * & 0 \end{pmatrix}, \quad (92)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & 0 & a_{31}x_1 & 0 & 0 & 0 \\ * & 0 & 0 & b_{42}x_2 + b_{44}x_4 & 0 & b_{42}x_2x_5 + b_{44}x_4x_5 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & e_{65}x_5 \\ * & * & * & * & * & 0 \end{pmatrix}. \quad (93)$$

Now by means of (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{a_{31}x_1}{p_{13}} + \frac{b_{42}x_2 + b_{44}x_4}{p_{24}} + \frac{e_{65}x_5}{p_{56}}, \quad H_2 = 1/2((\frac{a_{31}x_1}{p_{13}})^2 + (\frac{b_{42}x_2 + b_{44}x_4}{p_{24}})^2 + (\frac{e_{65}x_5}{p_{56}})^2),$$

$$H_3 = 1/3((\frac{a_{31}x_1}{p_{13}})^3 + (\frac{b_{42}x_2 + b_{44}x_4}{p_{24}})^3 + (\frac{e_{65}x_5}{p_{56}})^3). \quad (94)$$

**Lie group  $\mathbf{A}_{6,26}$ :**

With substituting  $P$ ,  $P'$  and vierbein matrix in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $\mathbf{A}_{6,26}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & p_{14} & 0 & 0 \\ * & 0 & p_{23} & 0 & 0 & 0 \\ * & * & 0 & \frac{c_{44}p_{14}}{a_{44}} + p_{14}x_2 & p_{35} & \frac{d_{64}p_{35}}{d_{54}} \\ * & * & * & 0 & \frac{d_{54}p_{14}}{a_{44}} & \frac{d_{64}p_{14}}{a_{44}} \\ * & * & * & * & 0 & p_{56} \\ * & * & * & * & * & 0 \end{pmatrix}, \quad (95)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & 0 & 0 & a_{44}x_4 & 0 & 0 \\ * & 0 & b_{32}x_2 & 0 & 0 & 0 \\ * & * & 0 & c_{44}x_4 + a_{44}x_2x_4 & 0 & 0 \\ * & * & * & 0 & d_{54}x_4 & d_{64}x_4 \\ * & * & * & * & 0 & e_{65}x_5 - \frac{d_{54}e_{65}x_6}{d_{64}} \\ * & * & * & * & * & 0 \end{pmatrix}. \quad (96)$$

Now by means of (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{b_{32}x_2}{p_{23}} + \frac{a_{44}x_4}{p_{14}} + \frac{e_{65}(d_{64}x_5 - d_{54}x_6)}{d_{64}p_{56}}, \quad H_2 = 1/2((\frac{b_{32}x_2}{p_{23}})^2 + (\frac{a_{44}x_4}{p_{14}})^2 + (\frac{e_{65}(d_{64}x_5 - d_{54}x_6)}{d_{64}p_{56}})^2),$$

$$H_3 = 1/3((\frac{b_{32}x_2}{p_{23}})^3 + (\frac{a_{44}x_4}{p_{14}})^3 + (\frac{e_{65}(d_{64}x_5 - d_{54}x_6)}{d_{64}p_{56}})^3). \quad (97)$$

**Lie group  $\mathbf{A}_{6,27}$ :**

With substituting  $P$ ,  $P'$  and vierbein matrix in (12) one can obtain the compatible Poisson structures  $\mathbf{P}$  and  $\mathbf{P}'$  on the Lie group  $\mathbf{A}_{6,27}$  as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & p_{12} & p_{13} & 0 & 0 & 0 \\ * & 0 & 0 & p_{24} & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & p_{56} \\ * & * & * & * & * & 0 \end{pmatrix}, \quad (98)$$

$$\mathbf{P}' = \begin{pmatrix} 0 & 0 & a_{33}x_3 & 0 & 0 & 0 \\ * & 0 & 0 & b_{44}x_4 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & e_{65}x_5 + e_{66}x_6 \\ * & * & * & * & * & 0 \end{pmatrix}. \quad (99)$$

Now by means of (34), the integrals of motion could be found for this Lie group as follows:

$$H_1 = \frac{a_{33}x_3}{p_{13}} + \frac{b_{44}x_4}{p_{24}} + \frac{e_{65}x_5 + e_{66}x_6}{p_{56}}, \quad H_2 = 1/2((\frac{a_{33}x_3}{p_{13}})^2 + (\frac{b_{44}x_4}{p_{24}})^2 + (\frac{e_{65}x_5 + e_{66}x_6}{p_{56}})^2),$$

$$H_3 = 1/3((\frac{a_{33}x_3}{p_{13}})^3 + (\frac{b_{44}x_4}{p_{24}})^3 + (\frac{e_{65}x_5 + e_{66}x_6}{p_{56}})^3). \quad (100)$$

## 7 Concluding remarks

Using the adjoint representation of the Lie algebra we have given a new method for calculation of compatible Poisson structures on four and nilpotent six dimensional symplectic real Lie algebras. Also by use of Magri-Morosi's theorem we have obtained new bi-Hamiltonian systems with these Lie groups as phase spaces. As an open problem, one can obtain another set of compatible Poisson structures by setting  $P'$  as a second order functions of Lie group parameters and in this way new bi-Hamiltonian systems can be obtained.

**Appendix A:** The list of symplectic four dimensional real Lie algebras [14]

$\mathfrak{g}$	Non-zero commutation relations	$\mathfrak{g}$	Non-zero commutation relations
$A_{4,1}$	$[\mathbf{e}_2, \mathbf{e}_4] = \mathbf{e}_1, [\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_2$	$A_{4,2}^{-1}$	$[\mathbf{e}_1, \mathbf{e}_4] = -\mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_4] = \mathbf{e}_2, [\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_2 + \mathbf{e}_3$
$A_{4,3}$	$[\mathbf{e}_1, \mathbf{e}_4] = \mathbf{e}_1, [\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_2$	$A_{4,5}^{-1,-1}$	$[\mathbf{e}_1, \mathbf{e}_4] = \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_4] = -\mathbf{e}_2, [\mathbf{e}_3, \mathbf{e}_4] = -\mathbf{e}_3$
$A_{4,5}^{-1,b}$	$[\mathbf{e}_1, \mathbf{e}_4] = \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_4] = -\mathbf{e}_2, [\mathbf{e}_3, \mathbf{e}_4] = b \mathbf{e}_3$	$A_{4,5}^{-1}$	$[\mathbf{e}_1, \mathbf{e}_4] = \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_4] = a \mathbf{e}_2, [\mathbf{e}_3, \mathbf{e}_4] = -\mathbf{e}_3$
$A_{4,5}^{a,-a}$	$[\mathbf{e}_1, \mathbf{e}_4] = \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_4] = a \mathbf{e}_2, [\mathbf{e}_3, \mathbf{e}_4] = -a \mathbf{e}_3$	$A_{4,6}^{a,0}$	$[\mathbf{e}_1, \mathbf{e}_4] = a \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_4] = -\mathbf{e}_3, [\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_2$
$A_{4,7}$	$[\mathbf{e}_1, \mathbf{e}_4] = 2\mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_4] = \mathbf{e}_2$ $[\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_2 + \mathbf{e}_3$	$A_{4,9}^0$	$[\mathbf{e}_1, \mathbf{e}_4] = \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_4] = \mathbf{e}_2$
$A_{4,9}^1$	$[\mathbf{e}_1, \mathbf{e}_4] = 2\mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_4] = \mathbf{e}_2$ $[\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_3$	$A_{4,9}^{-1/2}$	$[\mathbf{e}_1, \mathbf{e}_4] = 1/2 \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_4] = \mathbf{e}_2$ $[\mathbf{e}_3, \mathbf{e}_4] = -1/2 \mathbf{e}_3$
$A_{4,11}^b$	$[\mathbf{e}_1, \mathbf{e}_4] = 2a \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1$ $[\mathbf{e}_2, \mathbf{e}_4] = a \mathbf{e}_2 - \mathbf{e}_3, [\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_2 + a \mathbf{e}_3$	$A_{4,9}^b$	$[\mathbf{e}_1, \mathbf{e}_4] = (1+b) \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_4] = \mathbf{e}_2$ $[\mathbf{e}_3, \mathbf{e}_4] = b \mathbf{e}_3$
$A_2 \oplus A_2$	$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_2, [\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_4$	$A_{4,12}$	$[\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1, [\mathbf{e}_1, \mathbf{e}_4] = -\mathbf{e}_2, [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_2$ $[\mathbf{e}_2, \mathbf{e}_4] = \mathbf{e}_1$
$VI_0 \oplus R$	$[\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_2, [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1$	$III \oplus R$	$[\mathbf{e}_1, \mathbf{e}_2] = -\mathbf{e}_2 - \mathbf{e}_3, [\mathbf{e}_1, \mathbf{e}_3] = -\mathbf{e}_2 - \mathbf{e}_3$
$VII_0 \oplus R$	$[\mathbf{e}_1, \mathbf{e}_3] = -\mathbf{e}_2, [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1$	$II \oplus R$	$[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1$

**Appendix B:** The list of symplectic nilpotent 6-dimensional real Lie algebras and their vierbein matrices

$\mathfrak{g}$	Non-zero commutation relations	vierbein matrix
$A_{6,1}$	$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3$ $[\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_4$ $[\mathbf{e}_1, \mathbf{e}_5] = \mathbf{e}_6$	$\begin{pmatrix} 1 & 0 & x_2 & x_3 & 0 & x_5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$A_{6,2}$	$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3$ $[\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_4$ $[\mathbf{e}_1, \mathbf{e}_4] = \mathbf{e}_5$ $[\mathbf{e}_1, \mathbf{e}_5] = \mathbf{e}_6$	$\begin{pmatrix} 1 & 0 & x_2 & x_3 & x_4 & x_5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$A_{6,3}$	$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_6$ $[\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_4$ $[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_5$	$\begin{pmatrix} 1 & 0 & 0 & x_3 & 0 & x_2 \\ 0 & 1 & 0 & 0 & x_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$A_{6,4}$	$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_5$ $[\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_6$ $[\mathbf{e}_2, \mathbf{e}_4] = \mathbf{e}_6$	$\begin{pmatrix} 1 & 0 & 0 & 0 & x_2 & x_3 \\ 0 & 1 & 0 & 0 & 0 & x_4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

[illegible]



<b>g</b>	Non-zero commutation relations	vierbein matrix
$A_{6,19}$	$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3$	$\begin{pmatrix} 1 & 0 & x_2 & x_3 & x_4 & -x_2^2/2 + x_5 \\ 0 & 1 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
	$[\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_4$	
	$[\mathbf{e}_1, \mathbf{e}_4] = \mathbf{e}_5$	
	$[\mathbf{e}_1, \mathbf{e}_5] = \mathbf{e}_6$	
	$[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_6$	
$A_{6,20}$	$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3$	$\begin{pmatrix} 1 & 0 & x_2 & x_3 & -x_2^2/2 + x_4 & x_5 \\ 0 & 1 & 0 & 0 & x_3 & x_4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
	$[\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_4$	
	$[\mathbf{e}_1, \mathbf{e}_4] = \mathbf{e}_5$	
	$[\mathbf{e}_1, \mathbf{e}_5] = \mathbf{e}_6$	
	$[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_5$	
$A_{6,24}$	$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3$	$\begin{pmatrix} 1 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$A_{6,25}$	$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3$	$\begin{pmatrix} 1 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & x_5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
	$[\mathbf{e}_4, \mathbf{e}_5] = \mathbf{e}_6$	
$A_{6,26}$	$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3$	$\begin{pmatrix} 1 & 0 & x_2 & x_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
	$[\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_4$	
$A_{6,27}$	$[\mathbf{e}_3, \mathbf{e}_5] = \mathbf{e}_1$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ x_5 & 0 & 1 & 0 & 0 & 0 \\ 0 & x_5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
	$[\mathbf{e}_4, \mathbf{e}_5] = \mathbf{e}_2$	
$A_{6,28}$	$[\mathbf{e}_2, \mathbf{e}_5] = \mathbf{e}_1$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ x_5 & 1 & 0 & 0 & 0 & 0 \\ x_5^2/2 & x_5 & 1 & 0 & 0 & 0 \\ x_5^3/6 & x_5^2/2 & x_5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
	$[\mathbf{e}_3, \mathbf{e}_5] = \mathbf{e}_2$	
	$[\mathbf{e}_4, \mathbf{e}_5] = \mathbf{e}_3$	
$A_{6,31}$	$[\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_1$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ x_5 & 1 & 0 & 0 & 0 & 0 \\ x_4 + x_5^2/2 & x_5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
	$[\mathbf{e}_2, \mathbf{e}_5] = \mathbf{e}_1$	
	$[\mathbf{e}_3, \mathbf{e}_5] = \mathbf{e}_2$	
$A_{6,32}$	$[\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_1$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ x_5 & 1 & 0 & 0 & 0 & 0 \\ x_4 + x_5^2/2 & x_5 & 1 & 0 & 0 & 0 \\ x_5^3/6 & x_5^2/2 & x_5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
	$[\mathbf{e}_2, \mathbf{e}_5] = \mathbf{e}_1$	
	$[\mathbf{e}_3, \mathbf{e}_5] = \mathbf{e}_2$	
	$[\mathbf{e}_4, \mathbf{e}_5] = \mathbf{e}_3$	

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